

ANALYTICAL SOLUTION OF THE WEIGHTED FERMAT-TORRICELLI PROBLEM FOR TETRAHEDRA: THE CASE OF TWO PAIRS OF EQUAL WEIGHTS

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ABSTRACT. The weighted Fermat-Torricelli problem for four non-collinear and non-coplanar points in \mathbb{R}^3 states that:

Given four non-collinear and non-coplanar points A_1, A_2, A_3, A_4 and a positive real number (weight) B_i which correspond to each point A_i , for $i = 1, 2, 3, 4$, find a fifth point such that the sum of the weighted distances to these four points is minimized. We present an analytical solution for the weighted Fermat-Torricelli problem for tetrahedra in \mathbb{R}^3 for the case of two pairs of equal weights.

1. INTRODUCTION

Let A_1, A_2, A_3, A_4 be four non-collinear and non-coplanar points and a positive real number (weight) B_i correspond to each point A_i , for $i = 1, 2, 3, 4$.

The weighted Fermat-Torricelli problem for four non-collinear points and non-coplanar points in \mathbb{R}^3 states that:

Problem 1. *Find a unique (fifth) point $A_0 \in \mathbb{R}^3$, which minimizes*

$$f(X) = \sum_{i=1}^4 B_i \|X - A_i\|,$$

where $\|\cdot\|$ denotes the Euclidean distance and $X \in \mathbb{R}^3$.

The existence and uniqueness of the weighted Fermat-Torricelli point and a complete characterization of the solution of the weighted Fermat-Torricelli problem for tetrahedra has been established in [4, Theorem 1.1, Reformulation 1.2 page 58, Theorem 8.5 page 76, 77]).

Theorem 1. [1],[4],[3] *Let there be given four non-collinear points and non-coplanar points $\{A_1, A_2, A_3, A_4\}$, $A_1, A_2, A_3, A_4 \in \mathbb{R}^3$ with corresponding positive weights B_1, B_2, B_3, B_4 .*

- (a) *The weighted Fermat-Torricelli point A_0 exists and is unique.*
- (b) *If for each point $A_i \in \{A_1, A_2, A_3, A_4\}$*

$$\left\| \sum_{j=1, j \neq i}^4 B_j \vec{u}(A_i, A_j) \right\| > B_i, \tag{1.1}$$

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for $i, j = 1, 2, 3$ holds, then

(b₁) the weighted Fermat-Torricelli point A_0 (weighted floating equilibrium point) does not belong to $\{A_1, A_2, A_3, A_4\}$ and

(b₂)

$$\sum_{i=1}^4 B_i \vec{u}(A_0, A_i) = \vec{0}, \quad (1.2)$$

where $\vec{u}(A_k, A_l)$ is the unit vector from A_k to A_l , for $k, l \in \{0, 1, 2, 3, 4\}$ (Weighted Floating Case).

(c) If there is a point $A_i \in \{A_1, A_2, A_3, A_4\}$ satisfying

$$\left\| \sum_{j=1, j \neq i}^4 B_j \vec{u}(A_i, A_j) \right\| \leq B_i, \quad (1.3)$$

then the weighted Fermat-Torricelli point A_0 (weighted absorbed point) coincides with the point A_i (Weighted Absorbed Case).

We consider the following open problem:

Problem 2. Find an analytic solution with respect to the weighted Fermat-Torricelli problem for tetrahedra in \mathbb{R}^3 , such that the corresponding weighted Fermat-Torricelli point is not any of the given points.

In this paper, we present an analytical solution for the weighted Fermat-Torricelli problem for regular tetrahedra in \mathbb{R}^3 for $B_1 > B_4$, $B_1 = B_2$ and $B_3 = B_4$, by expressing the objective function as a function of the linear segment which is formed by the middle point of the common perpendicular A_1A_2 and A_3A_4 , and the corresponding weighted Fermat-Torricelli point A_0 (Section 2, Theorem 2). It is worth mentioning that this analytical solution of the weighted Fermat-Torricelli problem for a regular tetrahedron is a generalization of the analytical solution of the weighted Fermat-Torricelli point of a quadrangle (tetragon) in \mathbb{R}^2 (see in [10]).

By expressing the angles $\angle A_i A_0 A_j$ for $i, j = 1, 2, 3, 4$ for $i \neq j$ as a function of B_1 , B_4 and a and taking into account the invariance property of the weighted Fermat-Torricelli point (geometric plasticity) in \mathbb{R}^3 , we obtain an analytical solution for some tetrahedra having the same weights with the regular tetrahedron (Section 3, Theorem 3).

2. THE WEIGHTED FERMAT-TORRICELLI PROBLEM FOR REGULAR TETRAHEDRA:

THE CASE $B_1 = B_2$ AND $B_3 = B_4$.

We shall consider the weighted Fermat-Torricelli problem for a regular tetrahedron $A_1A_2A_3A_4$, for $B_1 > B_4$, $B_1 = B_2$ and $B_3 = B_4$.

We denote by a_{ij} the length of the linear segment A_iA_j , by $A_{12}A_{34}$ the common perpendicular of A_1A_2 and A_3A_4 where A_{12} is the middle point of A_1A_2 and A_{34} is the middle point of A_3A_4 , by A_0 the weighted Fermat-Torricelli point of $A_1A_2A_3A_4$ by O the middle point of $A_{12}A_{34}$ ($A_{12}O = A_{34}O$), by y the length of the linear segment OA_0 and α_{ikj} the angle $\angle A_i A_k A_j$ for $i, j, k = 0, 1, 2, 3, 4, i \neq j \neq k$. We set $a_{ij} \equiv a$, the edges of $A_1A_2A_3A_4$ (Fig. 1).

Problem 3. Given a regular tetrahedron $A_1A_2A_3A_4$ and a weight B_i which corresponds to the vertex A_i , for $i = 1, 2, 3, 4$, find a fifth point A_0 (weighted Fermat-Torricelli point) which minimizes the objective function

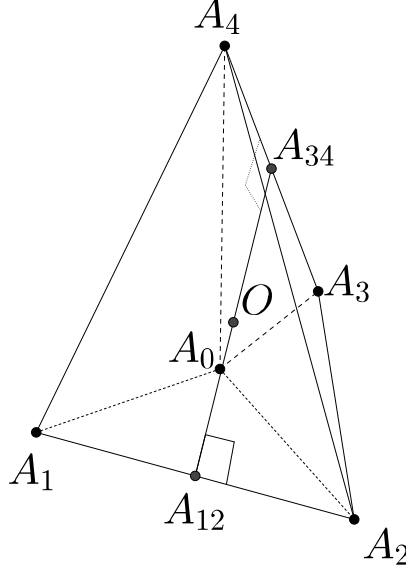


FIGURE 1. The weighted Fermat-Torricelli problem for a regular tetrahedron $B_1 = B_2$ and $B_3 = B_4$ for $B_1 > B_4$

$$f = B_1 a_{01} + B_2 a_{02} + B_3 a_{03} + B_4 a_{04} \quad (2.1)$$

for $B_1 > B_4$, $B_1 = B_2$ and $B_3 = B_4$.

We set

$$\begin{aligned} s \equiv & -a^6 B_1^{12} + 2a^6 B_1^{10} B_4^2 + a^6 B_1^8 B_4^4 - 4a^6 B_1^6 B_4^6 + a^6 B_1^4 B_4^8 + 2a^6 B_1^2 B_4^{10} - a^6 B_4^{12} + 2\sqrt{2} \\ & \sqrt{(a^{12} B_1^{22} B_4^2 - 8a^{12} B_1^{20} B_4^4 + 29a^{12} B_1^{18} B_4^6 - 64a^{12} B_1^{16} B_4^8 + 98a^{12} B_1^{14} B_4^{10} - 112a^{12} B_1^{12} B_4^{12} +} \\ & + 98a^{12} B_1^{10} B_4^{14} - 64a^{12} B_1^8 B_4^{16} + 29a^{12} B_1^6 B_4^{18} - 8a^{12} B_1^4 B_4^{20} + a^{12} B_1^2 B_4^{22})} \end{aligned} \quad (2.2)$$

and

$$t \equiv -\frac{a^4 B_1^4}{4s^{1/3}} + \frac{a^4 B_1^2 B_4^2}{2s^{1/3}} - \frac{a^4 B_4^4}{4s^{1/3}} - \frac{s^{1/3}}{4(B_1^4 - 2B_1^2 B_4^2 + B_4^4)}. \quad (2.3)$$

Theorem 2. *The location of the weighted Fermat-Torricelli point A_0 of $A_1 A_2 A_3 A_4$ for $B_1 = B_2$, $B_3 = B_4$ and $B_1 > B_4$ is given by:*

$$y = -\frac{\sqrt{t}}{2} + \frac{1}{2} \sqrt{\frac{a^4 B_1^4}{4s^{1/3}} - \frac{a^4 B_1^2 B_4^2}{2s^{1/3}} + \frac{a^4 B_4^4}{4s^{1/3}} + \frac{2(-8\sqrt{2}a^3 B_1^2 - 8\sqrt{2}a^3 B_4^2)}{\sqrt{t}(64B_1^2 - 64B_4^2)} + \frac{s^{1/3}}{4(B_1^4 - 2B_1^2 B_4^2 + B_4^4)}} \quad (2.4)$$

Proof of Theorem 2: Taking into account the symmetry of the weights $B_1 = B_4$ and $B_2 = B_3$ for $B_1 > B_4$ and the symmetries of the regular tetrahedron $A_1 A_2 A_3 A_4$ the objective function (2.13) of the weighted Fermat-Torricelli problem (Problem 3) could be reduced to an equivalent Problem: Find a point A_0 which belongs to the midperpendicular $A_{12}A_{34}$ of $A_1 A_2$ and $A_3 A_4$ and minimizes the objective function

$$\frac{f}{2} = B_1 a_{01} + B_4 a_{04}. \quad (2.5)$$

We express a_{01} , a_{02} , a_{03} and a_{04} as a function of y :

$$a_{01}^2 = \left(\frac{a}{2}\right)^2 + \left(\frac{a\sqrt{2}}{2} - y\right)^2, \quad (2.6)$$

$$a_{02}^2 = \left(\frac{a}{2}\right)^2 + \left(\frac{a\sqrt{2}}{2} - y\right)^2, \quad (2.7)$$

$$a_{03}^2 = \left(\frac{a}{2}\right)^2 + \left(\frac{a\sqrt{2}}{2} + y\right)^2, \quad (2.8)$$

$$a_{04}^2 = \left(\frac{a}{2}\right)^2 + \left(\frac{a\sqrt{2}}{2} + y\right)^2, \quad (2.9)$$

where the length of $A_{12}A_{34}$ is $\frac{a\sqrt{2}}{2}$.

By replacing (2.6) and (2.9) in (2.5) we get:

$$B_1 \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a\sqrt{2}}{2} - y\right)^2} + B_4 \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a\sqrt{2}}{2} + y\right)^2} \rightarrow \min. \quad (2.10)$$

By differentiating (2.10) with respect to y , and by squaring both parts of the derived equation, we get:

$$\frac{B_1^2 \left(\frac{a\sqrt{2}}{2} - y\right)^2}{\left(\frac{a}{2}\right)^2 + \left(\frac{a\sqrt{2}}{2} - y\right)^2} = \frac{B_4^2 \left(\frac{a\sqrt{2}}{2} + y\right)^2}{\left(\frac{a}{2}\right)^2 + \left(\frac{a\sqrt{2}}{2} + y\right)^2} \quad (2.11)$$

which yields

$$64y^4 (B_1^2 - B_4^2) - 8\sqrt{2}a^3 y (B_1^2 + B_4^2) + 3a^4 (B_1^2 - B_4^2) = 0. \quad (2.12)$$

By solving the fourth order equation (2.12) with respect to y , we derive two complex solutions and two real solutions (see in [6] for the general solution of a

fourth order equation with respect to y) which depend on B_1, B_4 and a . One of the two real solutions with respect to y is (2.4). The real solution (2.4) gives the location of the weighted Fermat-Torricelli point A_0 at the interior of $A_1A_2A_3A_4$ (see fig. 1). \square

We shall state the Complementary weighted Fermat-Torricelli problem for a regular tetrahedron ([2, pp. 358]), in order to explain the second real solution which have been obtained by (2.12) with respect to y (see also in [10] for the case of a quadrangle).

Problem 4. *Given a regular tetrahedron $A_1A_2A_3A_4$ and a weight B_i (a positive or negative real number) which corresponds to the vertex A_i , for $i = 1, 2, 3, 4$, find a fifth point A_0 (weighted Fermat-Torricelli point) which minimizes the objective function*

$$f = B_1a_{01} + B_2a_{02} + B_3a_{03} + B_4a_{04} \quad (2.13)$$

for $\|B_1\| > \|B_4\|$, $B_1 = B_2$ and $B_3 = B_4$.

Proposition 1. *The location of the complementary weighted Fermat-Torricelli point A'_0 (solution of Problem 4) of the regular tetrahedron $A_1A_2A_3A_4$ for $B_1 = B_2 < 0$, $B_3 = B_4 < 0$ and $\|B_1\| > \|B_4\|$ is the exactly same with the location of the corresponding weighted Fermat-Torricelli point A_0 of $A_1A_2A_3A_4$ for $B_1 = B_2 > 0$, $B_3 = B_4 > 0$ and $\|B_1\| > \|B_4\|$.*

Proof of Proposition 1: Taking into account theorem 2, for $B_1 = B_2 < 0$, $B_3 = B_4 < 0$ we derive:

$$\vec{B}_1 + \vec{B}_2 + \vec{B}_3 + \vec{B}_4 = \vec{0} \quad (2.14)$$

or

$$(-\vec{B}_1) + (-\vec{B}_2) + (-\vec{B}_3) + (-\vec{B}_4) = \vec{0}. \quad (2.15)$$

From (2.14) and (2.15), we derive that the complementary weighted Fermat-Torricelli point A'_0 coincides with the weighted Fermat-Torricelli point A_0 . We note that the vectors \vec{B}_i may change direction from A_i to A_0 , simultaneously, for $i = 1, 2, 3, 4$. \square

Proposition 2. *The location of the complementary weighted Fermat-Torricelli point A'_0 (solution of Problem 4) of the regular tetrahedron $A_1A_2A_3A_4$ for $B_1 = B_2 < 0$, $B_3 = B_4 > 0$ or $B_1 = B_2 > 0$, $B_3 = B_4 < 0$ and $\|B_1\| > \|B_4\|$ is given by:*

$$y = \frac{\sqrt{t}}{2} + \frac{1}{2} \sqrt{\frac{a^4 B_1^4}{4s^{1/3}} - \frac{a^4 B_1^2 B_4^2}{2s^{1/3}} + \frac{a^4 B_4^4}{4s^{1/3}} - \frac{2(-8\sqrt{2}a^3 B_1^2 - 8\sqrt{2}a^3 B_4^2)}{\sqrt{t}(64B_1^2 - 64B_4^2)} + \frac{s^{1/3}}{4(B_1^4 - 2B_1^2 B_4^2 + B_4^4)}} \quad (2.16)$$

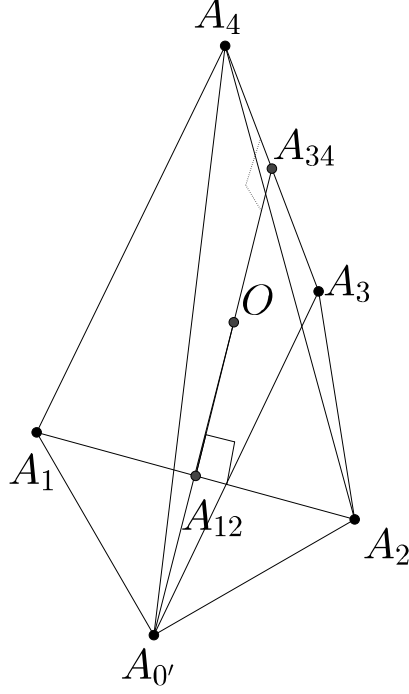


FIGURE 2. The complementary weighted Fermat-Torricelli point A'_0 of a regular tetrahedron $A_1A_2A_3A_4$ for $B_1 = B_2 > 0$ and $B_3 = B_4 < 0$ or $B_1 = B_2 < 0$ and $B_3 = B_4 > 0$ for $\|B_1\| > \|B_4\|$

Proof of Proposition 2: Considering (2.10) for $B_1 = B_2 < 0$, $B_3 = B_4 > 0$ or $B_1 = B_2 > 0$, $B_3 = B_4 < 0$ and $\|B_1\| > \|B_4\|$ and differentiating (2.10) with respect to $y \equiv OA'_0$, and by squaring both parts of the derived equation, we obtain (2.12) which is a fourth order equation with respect to y . The second real solution of y gives (2.16). Taking into account the real solution (2.16) and the weighted floating equilibrium condition $\vec{B}_1 + \vec{B}_2 + \vec{B}_3 + \vec{B}_4 = \vec{0}$ we obtain that the complementary weighted Fermat-Torricelli point A'_0 for $B_1 = B_2 < 0$, $B_3 = B_4 > 0$ coincides with the complementary weighted Fermat-Torricelli point A''_0 for $B_1 = B_2 > 0$, $B_3 = B_4 < 0$ (Fig. 2). From (2.16), we derive that the complementary A'_0 is located outside the regular tetrahedron $A_1A_2A_3A_4$ (Fig. 2).

□

Example 1. Given a regular tetrahedron $A_1A_2A_3A_4$ in \mathbb{R}^3 , $a = 1$, $B_1 = B_2 = 2.5$, $B_3 = B_4 = 1$ from (2.4) and (2.16) we get $y = 0.198358$ and $y = 0.539791$, respectively, with six digit precision. The weighted Fermat-Torricelli point A_0 and the complementary weighted Fermat-Torricelli point $A'_0 \equiv A_0$ for $B_1 = B_2 = -2.5$

and $B_3 = B_4 = -1$ corresponds to $y = 0.198358$. The complementary weighted Fermat-Torricelli point A'_0 for $B_1 = B_2 = -2.5$ and $B_3 = B_4 = 1$ or $B_1 = B_2 = 1.5$ and $B_3 = B_4 = -1$ lies outside the regular tetrahedron $A_1A_2A_3A_4$ and corresponds to $y = 0.539791 > \frac{A_{12}A_{34}}{2} = \frac{\sqrt{2}}{4}$.

We proceed by calculating the angles α_{i0j} , for $i, j = 0, 1, 2, 3, 4$.

Proposition 3. *The angles α_{i0j} , for $i, j = 0, 1, 2, 3, 4$, are given by:*

$$\alpha_{102} = \arccos \left(1 - \frac{a^2}{2 \left(\left(\frac{a}{2} \right)^2 + \left(\frac{\frac{a\sqrt{2}}{2}}{2} - y \right)^2 \right)} \right), \quad (2.17)$$

$$\alpha_{304} = \arccos \left(1 - \frac{a^2}{2 \left(\left(\frac{a}{2} \right)^2 + \left(\frac{\frac{a\sqrt{2}}{2}}{2} + y \right)^2 \right)} \right), \quad (2.18)$$

and

$$\alpha_{104} = \alpha_{203} = \alpha_{103} = \alpha_{204} = \arccos \frac{\left(\frac{\frac{a\sqrt{2}}{2}}{2} - y \right)^2 + \left(\frac{\frac{a\sqrt{2}}{2}}{2} + y \right)^2 - \frac{a^2}{2}}{2 \sqrt{\left(\frac{a}{2} \right)^2 + \left(\frac{\frac{a\sqrt{2}}{2}}{2} - y \right)^2} \sqrt{\left(\frac{a}{2} \right)^2 + \left(\frac{\frac{a\sqrt{2}}{2}}{2} + y \right)^2}}. \quad (2.19)$$

Proof of Proposition 3: Taking into account the cosine law in $\triangle A_1A_0A_2$, $\triangle A_3A_0A_4$, $\triangle A_1A_0A_4$, $\triangle A_2A_0A_4$, $\triangle A_1A_0A_3$, $\triangle A_2A_0A_4$, and (2.4), we obtain (2.17), (2.18) and (2.19), respectively. \square

Corollary 1. [5, Theorem 4.3, p. 102] *If $B_1 = B_2 = B_3 = B_4$, then*

$$\alpha_{i0j} = \arccos \left(-\frac{1}{3} \right), \quad (2.20)$$

for $i, j = 1, 2, 3, 4$ and $i \neq j$.

Proof. By setting $y = 0$ in (2.17), (2.18) and (2.19), we obtain (2.20). \square

3. THE WEIGHTED FERMAT-TORRICELLI PROBLEM FOR TETRAHEDRA IN THE THREE DIMENSIONAL EUCLIDEAN SPACE: THE CASE $B_1 = B_2$ AND $B_3 = B_4$.

We consider the following lemma which gives the invariance property (geometric plasticity) of the weighted Fermat-Torricelli point for a given tetrahedron $A'_1A'_2A'_3A'_4$ in \mathbb{R}^3 ([9, Appendix AII, pp. 851-853])

Lemma 1. [9, Appendix AII, pp. 851-853] *Let $A_1A_2A_3A_4$ be a regular tetrahedron in \mathbb{R}^3 and each vertex A_i has a non-negative weight B_i for $i = 1, 2, 3, 4$. Assume that the floating case of the weighted Fermat-Torricelli point A_0 occurs:*

$$\left\| \sum_{j=1, i \neq j}^4 B_j \vec{u}(A_i, A_j) \right\| > B_i. \quad (3.1)$$

If A_0 is connected with every vertex A_i for $i = 1, 2, 3, 4$ and a point A'_i is selected with corresponding non-negative weight B_i on the ray that is defined by the line segment A_0A_i and the tetrahedron $A'_1A'_2A'_3A'_4$ is constructed such that:

$$\left\| \sum_{j=1, i \neq j}^4 B_j \vec{u}(A'_i, A'_j) \right\| > B_i, \quad (3.2)$$

then the weighted Fermat-Torricelli point A'_0 of $A'_1A'_2A'_3A'_4$ is identical with A_0 .

We consider a tetrahedron $A_1A_2A_3A'_4$ which has as a base the equilateral triangle $\triangle A_1A_2A_3$ with side a and the vertex A'_4 is located on the ray A_0A_4 , with corresponding non-negative weights $B_1 = B_2$ at the vertices A_1, A_2 and $B_3 = B_4$ at the vertices A_3, A'_4 .

Assume that we choose B_1 and B_4 non negative weights which satisfy the inequalities (3.1), (3.2) and $B_1 > B_4$, which correspond to the weighted floating case of $A_1A_2A_3A_4$ and $A_1A_2A_3A'_4$.

We denote by $a_{i4'}$ the length of the linear segment $A_iA_{4'}$, the angle $\angle A_iA_kA_{4'}$ for $i, j, 0, 1, 2, 3, 4, i \neq j$, by $h_{0,12}$ the height of $\triangle A_0A_1A_2$ from A_0 to A_1A_2 , by α the dihedral angle between the planes $A_0A_1A_2$ and $A_3A_1A_2$ and by $\alpha_{g_{4'}}$ the dihedral angle between the planes $A_3A_1A_2$ and $A_{4'}A_1A_2$ and by A_0 the corresponding weighted Fermat-Torricelli point of the regular tetrahedron $A_1A_2A_3A_4$.

Theorem 3. *The location of the weighted Fermat-Torricelli point $A_{0'}$ of a tetrahedron $A_1A_2A_3A'_4$ which has as a base the equilateral triangle $\triangle A_1A_2A_3$ with side a and the vertex A'_4 is located on the ray A_0A_4 for $B_1 = B_2$ and $B_3 = B_4$, under the conditions (3.1), (3.2) and $B_1 > B_4$, is given by:*

$$a_{04'} = \sqrt{a_{20}^2 + a_{24'}^2 - 2a_{24'} \left(\sqrt{a_{02}^2 - h_{0,12}^2} \cos \alpha_{124'} + h_{0,12} \sin \alpha_{124'} \cos(\alpha_{g_{4'}} - \alpha) \right)} \quad (3.3)$$

where

$$\alpha = \arccos \frac{\frac{a_{02}^2 + a_{03}^2 - a_{03}^2}{2a_{23}} - \sqrt{a_2^2 - h_{0,12}^2} \cos \alpha_{123}}{h_{0,12} \sin \alpha_{123}} \quad (3.4)$$

and

$$h_{0,12} = \sqrt{\frac{4a_{01}^2 a_{02}^2 - (a_{01}^2 + a_{02}^2 - a_{12}^2)^2}{4a_{12}^2}}. \quad (3.5)$$

Proof of Theorem 3: From lemma 1, we get $A_{0'} \equiv A_0$. Therefore, we get the relations (3.3) and (3.4) from a generalization of the cosine law in \mathbb{R}^3 which has been introduced for tetrahedra in [8, Solution of Problem 1, Formulas (2.14) and (2.20), p. 116].

□

Remark 1. We may consider a tetrahedron $A_1A_2A_3A_4$ by placing A_3 on the ray defined by A_0A_3 and $A_{0''}$ is the corresponding weighted Fermat-Torricelli point. Taking into account lemma 1, we get $A_{0''} \equiv A_{0'} \equiv A_0$.

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